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SHORTER COMMUNICATIONS

TRANSIENT CONVECTIVE HEAT TRANSFER IN FLUIDS WITH VANISHING PRANDTL NUMBER: AN APPLICATION OF THE INTEGRAL METHOD

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NOMENCLATURE

- x, dimensionless distance along the wall;
- y, dimensionless distance perpendicular to the wall;
- t, dimensionless time;
- u, v, dimensionless fluid velocity components;
- T, dimensionless fluid temperature;
- U, reference velocity;
- L, characteristic length;
- Pe, Péclét number;
- T_0, constant wall temperature;
- q_0, constant wall heat flux;
- k, fluid thermal conductivity;
- q, wall heat flux;
- $f(x)$, dimensionless function related to wall temperature, equation (6);
- $\gamma(x)$ dimensionless function related to wall heat flux, equation (7);
- $X(x)$, defined in equation (18);
- $F(t)$, defined in equation (19);
- $\Phi(x)$, defined in equation (21);
- C_i, constants ($i = 1, 2, 3$);
- m, exponent in the wedge flow potential velocity;
- $\Gamma(x)$, Gamma function;
- κ, fluid thermal diffusivity;
- δ, dimensionless boundary-layer thickness;
- τ, defined in equation (36);
- α, defined in equation (35).

Subscripts

- 1, dimensional quantities;
- w, conditions at the wall.

Superscripts

- $'$, differentiation;
- $*$, steady state conditions.

INTRODUCTION

In a recent paper [1] Soliman and Chambrè presented a study of transient heat transfer to incompressible laminar boundary layer flows when the Prandtl number is zero. In this case the velocity boundary layer vanishes and the potential flow solution can be assumed throughout the thermal layer. By applying successively the von Mises and Fourier transformations, the second order partial differential energy equation was transformed in a first order equation. Analytical solutions when the wall is subjected to a variation in temperature or heat flux were obtained by solving the resulting equations with the method of characteristics.

The purpose of this paper is to show that an approximate solution of the same problem can be simply obtained using the von Kármán–Pohlhausen integral method. Assuming a thermal boundary layer of finite thickness and a cubic temperature profile, the energy equation is transformed in a first order equation which is then solved with the method of characteristics. Analytical solutions are obtained for arbitrary potential flows and variations in wall temperature and heat flux. A comparison with Soliman–Chambrè solution is presented for the case of wedge flows.

ANALYSIS

With the usual assumptions of negligible axial conduction and viscous dissipation, the energy equation for plane incompressible laminar boundary layer flows can be written:

$$\frac{\partial T_1}{\partial t_1} + u_1 \frac{\partial T_1}{\partial x_1} + v_1 \frac{\partial T_1}{\partial y_1} = \kappa \frac{\partial^2 T_1}{\partial y_1^2}. \quad (1)$$

In terms of the dimensionless variables:

$$t = \frac{t_1 U_1}{L}, \quad x = \frac{x_1}{L}, \quad y = \sqrt{Pe}\,\frac{y_1}{L}, \quad u = \frac{u_1}{U}, \quad v = \frac{v_1}{U}$$

$T = \frac{T_1}{T_0}$ for variation in wall temperature and

$T = \frac{T_1 k \sqrt{Pe}}{q_0 L}$ for variation in wall heat flux,

equation (1) becomes:

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + \sqrt{Pe}\, v\frac{\partial T}{\partial y} = \frac{\partial^2 T}{\partial y^2}. \tag{2}$$

The initial and boundary conditions are:

$$T(0, x, y) = 0 \qquad x > 0, \quad y > 0 \tag{3}$$

$$T(t, 0, y) = 0 \qquad t > 0, \quad y > 0 \tag{4}$$

$$T(t, x, \infty) = 0 \qquad t > 0, \quad x > 0 \tag{5}$$

and

$$T(t, x, 0) = f(x) \qquad t > 0, \quad x > 0 \tag{6}$$

or

$$\frac{\partial T}{\delta y}(t, x, 0) = \gamma(x) \qquad t > 0, \quad x > 0 \tag{7}$$

$f(x)$ and $\gamma(x)$ are two dimensionless functions of x that describe the prescribed wall temperature and heat flux distributions.

Assuming a finite thermal boundary layer of dimensionless thickness δ, equation (2) can be integrated with respect to y between 0 and δ, yielding:

$$\frac{\partial}{\partial t}\int_0^\delta T\,\mathrm{d}y + \frac{\partial}{\partial x}\int_0^\delta uT\,\mathrm{d}y = \left.\frac{\partial T}{\partial y}\right|_0^\delta \tag{8}$$

where use has been made of the incompressible continuity equation.

With the assumption of zero Prandtl number, the velocity u appearing in equation (8) is the potential flow solution and therefore depends upon the coordinate x only.

SOLUTION FOR A WALL TEMPERATURE VARIATION

According to von Kármán–Pohlhausen integral method [2], a cubic temperature profile is chosen:

$$T = f(x)\left(1 - \frac{3y}{2\delta} + \frac{y^3}{2\delta^3}\right) \tag{9}$$

satisfying the boundary conditions:

$$T(t, x, 0) = f(x) \tag{10}$$

$$T(t, x, \delta) = 0 \tag{11}$$

$$\frac{\partial T}{\partial y}(t, x, \delta) = 0 \tag{12}$$

$$\frac{\partial^2 T}{\partial y^2}(t, x, \delta) = 0. \tag{13}$$

Inserting expression (9) in equation (8) and performing the integrations, the following equation works out:

$$\frac{\partial \delta^2}{\partial t} + u(x)\frac{\partial \delta^2}{\partial x} = 8 - 2u'(x)\,\delta^2 - 2u(x)\frac{f'(x)}{f(x)} \tag{14}$$

with the initial and boundary conditions:

$$\delta(0, x) = 0 \tag{15}$$

$$\delta(t, 0) = 0. \tag{16}$$

A closed form analytical solution of this equation can be obtained using the method of characteristics. The characteristic system associated with equation (14) is [3]:

$$\mathrm{d}t = \frac{\mathrm{d}x}{u(x)} = \frac{\mathrm{d}\delta^2}{8 - 2\delta^2 u'(x) - 2\delta^2 u(x)\, f'(x)/f(x)}. \tag{17}$$

The equation of the characteristic lines is:

$$t = \int_0^x \frac{\mathrm{d}\eta}{u(\eta)} + C_1 = X(x) + C_1 \tag{18}$$

or, except for an arbitrary constant:

$$x - F(t) = C_2 \tag{19}$$

in the assumption that the expression $t = X(x)$ of the dividing characteristic is univocally invertible.

Equating the second and third term of equation (17) and integrating, yields:

$$\tfrac{1}{8}\delta^2 u^2(x) f^2(x) - \phi(x) f^2(x) + 2\int_0^x f'(\eta)\, f(\eta)\, \phi(\eta)\mathrm{d}\eta = c_3 \tag{20}$$

where:

$$\phi(x) = \int_0^x u(\eta)\,\mathrm{d}\eta, \qquad x \geqslant 0. \tag{21}$$

Applying the conditions (15) and (16), the following two expressions for the solution of equation (17) work out:

$$\delta = \frac{2\sqrt{2}}{f(x)\, u(x)}\left\{\phi(x)\, f^2(x) - \phi[x - F(t)]\, f[x - F(t)] - 2\int_{x - F(t)}^{x} f'(\eta) f(\eta)\, \phi(\eta)\mathrm{d}\eta\right\}^{\frac{1}{2}}, \qquad t \leqslant X(x) \tag{22}$$

$$\delta = \frac{2\sqrt{2}}{f(x)\, u(x)}\left\{\phi(x)\, f^2(x) - 2\int_0^x f'(\eta)\, f(\eta)\, \phi(\eta)\,\mathrm{d}\eta\right\}^{\frac{1}{2}} \qquad t \geqslant X(x). \tag{23}$$

Equation (22) is time-dependent and therefore describes the transient part of the process, whereas equation (23) describes the steady state.

Inserting the above relationships in expression (9), the dimensionless fluid temperature distribution can be obtained. In particular, the dimensional wall heat flux is given by:

$$q_w = -k\frac{\partial T_1}{\partial y_1} = -\frac{k(\sqrt{Pe})T_0}{L}\frac{\partial T}{\partial y} = \frac{3}{2}\frac{k(\sqrt{Pe})T_0}{L}\frac{f(x)}{\delta}. \tag{24}$$

VARIATION IN WALL HEAT FLUX

Following the previous procedure, the chosen temperature profile is:

$$T = \gamma(x)\delta\left(\frac{1}{3} - \frac{y}{\delta} + \frac{y^2}{\delta^2} - \frac{y^3}{3\delta^3}\right). \tag{25}$$

The resulting partial differential equation becomes:

$$\frac{\partial\delta^2}{\partial t} + u(x)\frac{\partial\delta^2}{\partial x} = 12 - \delta^2 u(x)\frac{\gamma'(x)}{\gamma(x)} - \delta^2 u'(x) \tag{26}$$

the solution of which is again given by two expressions:

$$\delta = \frac{2\sqrt{3}}{\sqrt{[\gamma(x)\,u(x)]}}\left\{\gamma(x)x - \gamma[x - F(t)]\,[x - F(t)] - \int_{x-F(t)}^{x}\gamma'(\eta)\mathrm{d}\eta\right\}^{\frac{1}{2}} \quad t \leqslant X(x) \tag{27}$$

$$\delta = \frac{2\sqrt{3}}{\sqrt{[\gamma(x)\,u(x)]}}\left\{\gamma(x)x - \int_0^x \gamma'(\eta)\,\mathrm{d}\eta\right\}^{\frac{1}{2}} \quad t \geqslant X(x). \tag{28}$$

The dimensional wall temperature is:

$$T_{1,w} = \frac{q_0 L}{k\sqrt{Pe}}T(t, x, 0) = \frac{q_0 L}{3k\sqrt{Pe}}\gamma(x)\,\delta. \tag{29}$$

COMPARISON WITH SOLIMAN–CHAMBRÈ SOLUTION

In order to compare the obtained solutions with those given in [1], the previous analysis has been applied to the problem of wedge flows. The dimensionless potential velocity is [2]:

$$u(x) = x^m; \quad 0 \leqslant m < 1, \quad x \geqslant 0 \tag{30}$$

The functions $X(x)$ and $F(t)$ become:

$$X(x) = x^{1-m}/(1 - m) \quad x \geqslant 0 \tag{31}$$

$$F(t) = [t(1 - m)]^{1/(1-m)} \quad t \geqslant 0. \tag{32}$$

The case $m = 1$ has been excluded since the integral in equation (18) diverges at the lower limit. The prescribed wall temperature and heat flux are simply assumed as spatially uniform, namely:

$$f(x) = \gamma(x) = 1. \tag{33}$$

In the case of a temperature variation, the ratio of the transient wall heat flux to its steady state value is obtained by equations (22)–(24):

$$\begin{aligned} q_{1,w}/q^*_{1,w} &= 1/(1 - \alpha^{m+1})^{\frac{1}{2}} \quad & \tau \leqslant 1/(1 - m) \\ q_{1,w}/q^*_{1,w} &= 1 \quad & \tau \geqslant 1/(1 - m) \end{aligned} \tag{34}$$

where:

$$\alpha = 1 - [\tau(1 - m)]^{1/(1-m)} \tag{35}$$

$$\tau = tu(x)/x = tx^{m-1} \tag{36}$$

and the steady state wall heat flux is given by equations (23) and (24)

$$q^*_{1,w} = \frac{3}{4\sqrt{2}}\frac{kT_0}{L}[Pe(m + 1)]^{\frac{1}{2}}\Big/\frac{x_1}{L}\Big/^{(m-1)/2}. \tag{37}$$

For a variation in the wall heat flux, the ratio of the transient wall temperature to its steady state value is given by equations (27)–(29):

$$\begin{aligned} T_{1,w}/T^*_{1,w} &= (1 - m)^{\frac{1}{2}(1-m)}\ \tau^{\frac{1}{2}(1-m)} \quad & \tau \leqslant 1/(1 - m) \\ T_{1,w}/T^*_{1,w} &= 1 \quad & \tau \geqslant 1/(1 - m) \end{aligned} \tag{38}$$

where $T^*_{1,w}$ is given by equations (28) and (29):

$$T^*_{1,w} = \frac{2\sqrt{3}}{3}\frac{q_0 L}{k\sqrt{Pe}}\left(\frac{x_1}{L}\right)^{(1-m)/2}. \tag{39}$$

The comparison between the obtained solutions and Soliman–Chambrè [1] exact ones is straightforward for the case of a variation in the wall temperature. The two solutions are almost identical, the only difference being the numerical factor $3/4\sqrt{2}$ instead of $1/\sqrt{\pi}$ in the steady state flux expression. For the variation in the heat flux, the comparison is shown in Fig. 1. Soliman–Chambrè solution (full line) is

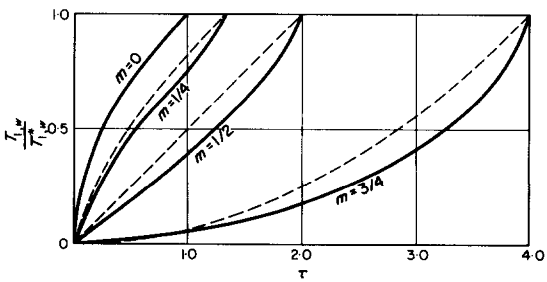

FIG. 1.

plotted together with the present one (dashed line) for $m = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$. As regard to the steady state value, the numerical factor $2(\sqrt{3})/3$ appears instead of

$$\Gamma\left(\frac{1}{1+m}\right)\Bigg/\left[(m+1)^{\frac{1}{2}}\,\Gamma\left(\frac{m+3}{2m+2}\right)\right].$$

CONCLUSIONS

The integral method has been applied to the study of transient heat transfer to boundary layer flows with zero Prandtl number. Analytical solutions have been obtained in a very simple way for arbitrary potential flows and have been compared with Soliman–Chambrè [1] exact solution in the particular case of wedge flows. The overall accuracy of the obtained solution seems satisfactory and gives further confidence in the possibility of application of the integral method to this kind of time-dependent problem. (A similar conclusion was drawn by Stewartson [4] in the problem of the impulsive motion of a flat plate in a viscous fluid.)

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ON THE PERTURBATION SOLUTION OF THE ICE–WATER LAYER PROBLEM

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NOMENCLATURE

- θ, departure from freezing temperature;
- ϕ, normalized temperature (θ/θ_c);
- X, distance from fixed cooled surface;
- x, normalized distance (X/X_c);
- t, time;
- τ, normalized time (t/t_c);
- θ_c, representative temperature difference;
- X_c, final layer thickness or "active zone" depth;
- t_c, characteristic time $(\rho L X_c^2/K\theta_c$ or periodic time);
- β, normalized ice thickness;
- C_p, specific heat at constant pressure;
- L, latent heat of fusion;
- ρ, ice density;
- K, thermal conductivity;
- Ste, Stefan number $(C_p\theta_c/L)$.

In a previous paper [1], a perturbation solution was developed for the formation of an ice layer at the edge of a semi-infinite domain of water, initially at the freezing point, and subject to a prescribed variation in surface temperature. In this form the problem was posed as the solution of the equations

$$\frac{\partial^2\phi}{\partial x^2} = Ste\frac{\partial\phi}{\partial\tau} \tag{1}$$

and

$$\frac{d\beta}{d\tau} = \left(\frac{\partial\phi}{\partial x}\right)_\beta \tag{2}$$

subject to suitable initial and boundary conditions: one of the boundary conditions has since been found to be incorrect.

A regular perturbation expansion in Ste was then used for both the temperature $\phi(x, \tau, Ste)$ and interface location $\beta(\tau, Ste)$. For two particular surface temperature variations (sinusoidal and power law) the interface expressions took particular forms which, it has since been noticed, are each of the more general form

$$\beta(\tau, Ste) = \beta(\tau, 0)\left[1 + \frac{Ste}{6}\phi^\circ(\tau) + 0(Ste^2)\right]$$

where $\phi^\circ(\tau)$ is the particular surface temperature variation. This is clearly a very simple and convenient result and therefore it is worthwhile examining the problem to see whether the result applies to any other surface temperature variations and to ascertain the extent of the error incurred through the use of an incorrect boundary condition.

Taking

$$\phi(x, \tau, Ste) = \sum_{p=0}^{\infty} Ste^p\,\phi_p(x, \tau), \tag{3}$$